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Propulsion by oscillating sheets and tubes in a viscous fluid

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Taylor's formula for the velocity of an oscillating sheet in a viscous fluid is extended to larger amplitudes of motion and his formula for the velocity of a rotating or oscillating filament is proved to be valid for amplitudes of motion small but greater than the filament radius. A simple formula is derived for the velocity of a sheet or filament in any general oscillatory motion of small amplitude. The forces and energy involved in oscillating tubes are calculated.

1. Introduction

Taylor (1951, 1952) has treated the problem of motion of an infinite sheet in sinusoidal motion in a viscous liquid at low Reynolds number and has shown that there is a non-zero velocity at which the thrust developed equals the drag. A similar treatment of an infinitely long cylinder oscillating transversely or in a rotating spiral leads him to a similar result, but in this case his choice of mathematical model makes his solution valid only for amplitudes of motion less than the radius of the cylinder.

Taylor's (1951) result for the sheet is extended here to larger amplitudes of motion and a different choice of mathematical model for the filament shows that Taylor's (1952) result for the filament is also valid for amplitudes of motion still small but greater than the filament radius and consistent with Hancock's (1953) results for very thin filaments for all amplitudes of motion.

2. The motion of an oscillating sheet

(*a*) Taylor (1951) showed that an infinite sheet in steady transverse sinusoidal motion in a plane progressive wave of small amplitude propels a viscous liquid forward in the direction of the wave at a non-zero velocity.

If we apply the same analysis to a sheet in a general transverse motion described by the lateral displacement,

$$y = \sum_n b_n \cos(k_n x - \sigma_n t + a_n) \tag{1}$$

we must relax Taylor's condition that the sheet is inextensible except when all the components of the wave have the same velocity.

However, if we write down the first-order terms for the fluid stream function and equate the normal component of fluid and sheet velocity to first and second order at the surface of the sheet, and, in place of the inextensibility conditions, we say that the velocity component parallel to the mean position of the sheet is periodic in both x and t to first order, then we obtain the fluid velocity at infinity

$$v = \sum_n \tfrac{1}{2} k_n \sigma_n b_n^2 + O(\sigma k^3 b^4), \tag{2}$$

provided no pairs of k and σ are identically equal.

The higher-order terms in the expression for v may be affected by the tangential motion (stretching or sliding) of the sheet, but not the first-order terms.

We may interpret this result physically as follows: the normal component of the sheet's motion will push liquid in the direction of the normal and the tangential component of the sheet's motion will drag liquid with it. If the sheet is inextensible and contains a wave of fixed shape of velocity V and wavelength L, while the arc length for the wavelength L is S, then at any point where the sheet is inclined at θ to its mean surface the normal component of velocity is $-V\sin\theta$, and the tangential component of velocity is $V(\cos\theta - S/L)$.

Resolving along the mean line of the sheet, the normal thrust tends to induce a velocity in the fluid which is the mean value of

$$V\sin^2\theta, \tag{3}$$

while the tangential drag tends to induce a velocity in the opposite direction which may for large amplitudes tend to infinity. We may then conclude that the fluid velocity has an upper bound V and no lower bound can be given for an inextensible sheet whose amplitude of motion is large.

Furthermore, for small values of θ,

$$V\sin^2\theta \doteqdot \sum_n \tfrac{1}{2} k_n \sigma_n b_n^2,$$

if σ_n/k_n is constant. Hence, for small amplitudes of motion, the velocity induced in the fluid is approximately that induced by the normal motion and the tangential effect is negligible.

(*b*) Taylor's (1951) rigorous treatment of viscous motion due to an inextensible sheet in sinusoidal motion for $bk < 1$ gives the fluid velocity at infinity

$$v = \tfrac{1}{2}\sigma b^2 k\left(1 - \tfrac{19}{16} b^2k^2 + \tfrac{41}{32} b^4k^4 - \tfrac{16913}{12288} b^6k^6 + \ldots\right), \tag{4}$$

on adding two more terms to Taylor's result.

This series is convergent for $bk < 1$ and may be expected on physical grounds to have an analytic continuation for $bk > 1$.

On using Shanks's (1955) e_1 operation repeatedly to improve the convergence of the series, we obtain the following values:

b^2k^2	$\frac{1}{4}$	$\frac{1}{2}$	1	2
vk/σ	0·0958	0·153	0·215	0·26

while for the range $0 < bk < 1$ these values may be fitted reasonably well by the formula

$$v = \frac{\sigma}{k}\frac{0{\cdot}518 b^2k^2}{1 + 1{\cdot}406 b^2k^2},$$

or
$$v = \frac{\sigma}{k}\frac{0\cdot508b^2k^2 - 0\cdot030b^4k^4}{1-1\cdot224b^2k^2}, \tag{5}$$

both formulae agreeing within a few per cent for $b^2k^2 < 2$, but being quite unreliable for larger amplitudes. These formulae may be compared with Hancock's (1953) formulae for plane and spiral waves in a thin filament.

(*c*) Since any periodic motion with finite oscillation may be analysed into a Fourier series of discrete terms, the displacement given by equation (1) represents any periodic motion, and furthermore if y is as given in equation (1) then

$$\mathrm{Av}\left(-\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right) = \sum_n \tfrac{1}{2}k_n\sigma_n b_n^2,$$

where the average is taken over both x and t. This is the liquid velocity given in equation (2). Hence we may conclude that, for any general oscillatory motion of a sheet in which the slope of the sheet is small, the first approximation to the velocity of the liquid relative to the sheet is

$$v = \mathrm{Av}\left(-\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right). \tag{6}$$

This formula may also be obtained for an oscillating filament from Hancock's (1953) vector formula. The corrected statement of the condition for finite energy in his formula (49) is

$$\lim_{s\to\infty}\frac{1}{2s}\int_{-s}^{s}\{f(x)\,\mathbf{t}+g(x)\,\mathbf{n}+h(x)\,\mathbf{b}\}\,ds = 0, \tag{7}$$

where s is the arc length replacing the linear dimension x and similarly in his formula (51), while his formula (47) gives the surface velocity of the filament, which need not be in the direction of the filament but any direction, so his formula (50) may be generalized to

$$-2f(x)\,\mathbf{t}-g(x)\,\mathbf{n}-h(x)\,\mathbf{b}+\mathbf{V} = \mathbf{U}, \tag{8}$$

where $\mathbf{U}$ is the velocity of the filament and $\mathbf{V}$ is the fluid velocity at infinity. Hence

$$f(x) = \tfrac{1}{2}\mathbf{t}\,.\,(\mathbf{V}-\mathbf{U}), \quad g(x) = \mathbf{n}\,.\,(\mathbf{V}-\mathbf{U}),$$

and
$$h(x) = \mathbf{b}\,.\,(\mathbf{V}-\mathbf{U}).$$

Substituting in the energy integral,

$$\lim_{s\to\infty}\frac{1}{2s}\int_{-s}^{s}[\mathbf{V}-\mathbf{U}-\tfrac{1}{2}\mathbf{t}\,.\,(\mathbf{V}-\mathbf{U})\,\mathbf{t}]\,ds = 0. \tag{9}$$

Hancock (1953) also concludes in § 3 that, apart from small end effects, finite filaments travel at the same speed as infinite filaments, so the above formula will also hold for a finite filament with the limits of integration changed to the ends of the filament.

For small amplitudes of motion, suppose the filament is close to the x-axis and oscillating transversely in both the y- and z-directions, then

$$\mathbf{U} = \left(0, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right).$$

Also $\mathbf{V}$ and $\mathbf{t}$ will be close to the x-direction and $ds \doteqdot dx$ so

$$\mathbf{t}.(\mathbf{V}-\mathbf{U}) \doteqdot V - \frac{\partial y}{\partial x}\frac{\partial y}{\partial t} - \frac{\partial z}{\partial x}\frac{\partial z}{\partial t}.$$

Substituting in the x-component of equation (9) we find

$$V = -\frac{1}{l}\int_0^l \left(\frac{\partial y}{\partial x}\frac{\partial y}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial t}\right) dx \tag{10}$$

for a filament oscillating with small amplitude of motion.

3. The motion of a viscous fluid round a rotating helix

Let the helix consist of a cylinder of radius a twisted into a helix of radius b and wavelength $2\pi/k$, rotated with frequency $\omega/2\pi$, so the equation of the arc of the helix is

$$x = b\cos(\omega t - kz), \quad y = b\sin(\omega t - kz)$$

for a helix rotating about the z-axis.

Taylor (1952) gives the velocity of propagation of the helix through a viscous liquid and an incorrect expression for the couple required to maintain this motion when the surface of the cylinder does not rotate, and points out that his analysis is valid only for small values of kb and for $b < a$, while it is exact for all values of ka. The reason for this is that, in the mathematical model used to represent the motion of the fluid, his functions would have singularities in the fluid for $b > a$.

To avoid this difficulty we use a skew co-ordinate system with the centre of the cylinder as the origin of cylindrical (r, θ) co-ordinates and z measured along the axis of the helix. The co-ordinate transformation from rectangular (x, y, z) co-ordinates is

$$x = b\cos(\theta-\theta') + r\cos\theta, \quad y = b\sin(\theta-\theta') + r\sin\theta, \quad z = \zeta,$$

where $\theta' = \theta + k\zeta - \omega t$.

The differential equations for the fluid velocity are then put in tensor notation, and then written out in terms of the skew co-ordinates, and a solution for the pressure p and velocity components u_r, u_θ, u_z is sought in the form

$$\begin{aligned} p &= \mu k(A_1\sin\theta' + bkA_2\sin 2\theta' + \ldots), \\ u_r &= B_1\sin\theta' + bkB_2\sin 2\theta' + \ldots, \\ u_\theta &= C_0 + C_1\cos\theta' + bkC_2\cos 2\theta' + \ldots, \\ u_z &= D_0 + D_1\cos\theta' + bkD_2\cos 2\theta' + \ldots, \end{aligned}$$

where the $ABCD$'s are functions of r only and are bounded for $a \leqslant r < \infty$ with the boundary condition on the surface of the cylinder $r = a$ that the fluid velocity

$$u_r = b\omega\sin\theta', \quad u_\theta = a\Omega + b\omega\cos\theta', \quad u_z = 0,$$

which is the velocity of the surface of the helix, which rotates with angular velocity ω while its surface skin rotates with angular velocity Ω ($\Omega = \omega$ for a rigid helix and $\Omega = 0$ for the case considered by Taylor (1952)).

On substituting in the differential equations we find that the lowest-order terms in bk are

$$A_1 = a_1 K_1(kr)/e_1,$$
$$B_1 = [b_1 K_0(kr) + c_1 K_2(kr) + a_1 kr K_1(kr)]/2e_1,$$
$$C_1 = [b_1 K_0(kr) - c_1 K_2(kr)]/2e_1,$$
$$D_1 = [d_1 K_1(kr) - a_1 kr K_0(kr)]/2e_1,$$

where the K's are modified Bessel functions while a_1, b_1, c_1, d_1 and e_1 are constant.

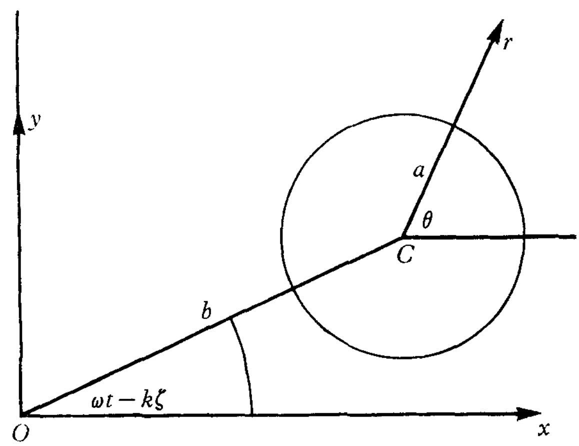

FIGURE 1. Cross-section of the helix.

From the boundary and divergence conditions

$$C_0 = a^2\Omega/r \quad \text{and} \quad D_0 = 0$$

to first order in kb and

$$a_1 = 2b\omega K_1(2K_1 + uK_0)/u,$$
$$b_1 = b\omega[u(K_1^2 - 2K_0^2) - 2K_1K_0 + 4K_1^2/u],$$
$$c_1 = -b\omega K_1^2 u,$$
$$d_1 = b\omega K_0(2K_1 + uK_0),$$
$$e_1 = (K_1 + uK_0)(K_1^2 - K_0^2) + 2K_0K_1^2/u,$$

where

$$u = ka, \quad K_0 = K_0(ka), \quad K_1 = K_1(ka),$$

while to second order in kb

$$C_0 = \frac{1}{r}\left\{a^2\Omega + \frac{b^2\omega}{2e_1} K_1[u^2(K_1^2 - K_0^2) + 4K_1^2]\right\} - \frac{bk}{4e_1}(a_1 + b_1 - c_1) K_1(kr), \tag{11}$$

and

$$D_0 = \frac{\omega k b^2}{2e_1}(2K_1 + uK_0)(K_0^2 - K_1^2 + 2K_0K_1/u) + \frac{bk}{4e_1}\{a_1 kr K_1(kr) - 2(a_1 + d_1) K_0(kr)\}. \tag{12}$$

The velocity of the fluid at infinity produced by rotating the helix is the constant term in D_0. This is the same result as derived by Taylor (1952), valid for all ka

and a first-order approximation for small kb, but this analysis using spiral co-ordinates shows that the restriction $b < a$ is not necessary.

We now proceed to list some further results of the theory. If ka is small the fluid velocity at infinity is

$$\frac{\omega k b^2}{2K_0+1}\left(2K_0 - 1 + \frac{k^2a^2}{2}\frac{8K_0^2-2K_0+3}{2K_0+1} + O(ka)^3\right), \tag{13}$$

and the circulatory velocity of the fluid for large r is

$$a^2\Omega/r + 2b^2\omega/(2K_0+1)r + O(ka), \tag{14}$$

while if ka is large the fluid velocity at infinity is

$$\frac{\omega k b^2}{2}\left\{1 + \frac{1}{ka} + O\left(\frac{1}{(ka)^2}\right)\right\}, \tag{15}$$

and the circulatory velocity of the fluid for large r is

$$a^2\Omega/r + (b^2\omega/2r)(ka + \tfrac{3}{2} + O(ka)^{-1}). \tag{16}$$

The couple required to produce this motion is obtainable from the resultant thrust and couple of the tractions on the skin of a section of the helix. If the contact between the skin and core of a circular section of the helix is smooth, then the only force between the skin and core will be a radial pressure, so the only action on the core will be a force through the centre of the section. The traction will also produce a couple acting only on the skin.

The only action at C in figure 1 is a force whose line of action is perpendicular to OC and opposes the rotation. Its magnitude is

$$\pi\mu b\omega u K_1(uK_1^2 - uK_0^2 + 2K_0K_1 + 8K_1^2/u)/e_1. \tag{17}$$

The couple on the skin of the helix is

$$-\pi\mu\{4a^2\Omega - b^2\omega u^2 K_1(K_0^2 - K_1^2 + 2K_0K_1/u)/e_1\}, \tag{18}$$

and the resultant couple about the axis, O, of the helix is

$$-4\pi\mu\{a^2\Omega + b^2\omega K_1(u^2K_1^2 - u^2K_0^2 + 4K_1^2)/2e_1\} = -4\pi\mu(rC_0), \tag{19}$$

and the energy output per unit length is

$$\pi\mu\{4a^2\Omega^2 - b^2\Omega\omega u^2K_1(K_0^2 - K_1^2 + 2K_0K_1/u)e_1 + b^2\omega^2uK_1(uK_1^2 - uK_0^2 + 2K_0K_1 + 8K_1^2/u)\}/e_1 \doteqdot 4\pi\mu b^2\omega^2/(K_0 + \tfrac{1}{2}) \tag{20}$$

for small ka. The case for $\Omega = 0$ was considered by Taylor (1952). He incorrectly used w_1' instead of v_1' in his equations (3.10) and (3.12), and the expression for the couple in his equation (3.13) should be

$$4\pi\mu b^2\kappa U/\{K_0(\kappa a) + \tfrac{1}{2}\} \quad \text{instead of} \quad -4\pi\mu a b^2\kappa^2 U\left\{\frac{K_0(\kappa a) - \frac{1}{2}}{K_0(\kappa a) + \frac{1}{2}}\right\}.$$

The former expression agrees with (19) above. The remainder of Taylor's paper is not affected by this error.

4. The motion of an oscillating tube in a viscous fluid

In the case of a tube oscillating in a plane through its axis a set of skew co-ordinates may be used

$$x = b\cos(k\zeta - \omega t) + r\cos\theta, \quad y = r\sin\theta, \quad z = \zeta$$

and the surface of the tube is $r = a$.

The hydrodynamic equations can be expressed in terms of these skew co-ordinates and solved for small kb while a may be as large or as small as we please. Apart from the energy output per unit length the results agree with Taylor's (1952).

If kb is small, the force exerted by the liquid on the tube is in the x-direction at right angles to the tube and in the opposite direction to the tube's transverse velocity and of magnitude

$$\pi\mu b\omega(8K_1^2 + 2uK_0K_1 + u^2(K_1^2 - K_0^2))\,(K_1/e_1)\sin(kz - \omega t), \tag{21}$$

where $u = ka$ and K_0 and K_1 are modified Bessel functions of argument ka.

If ka is small the transverse force is

$$4\pi\mu b\omega\sin(kz - \omega t)/(K_0 + \tfrac{1}{2})$$

and the average energy required per unit length to maintain the motion is

$$2\pi\mu b^2\omega^2/(K_0 + \tfrac{1}{2}). \tag{22}$$

The corresponding value given by Taylor (1952) is half this, owing to the omission of a factor $\frac{1}{2}$ before the quantities B and C in his equation (2.36). The consistency of the present result may be easily confirmed by comparison with the result for the helix for small ka. If the transverse velocity of the tube is

$$v = b\omega\sin(kz - \omega t),$$

then the force on the tube is

$$-4\pi\mu v/(K_0 + \tfrac{1}{2}).$$

If such a force acts about the axis at a distance b from the axis, it produces a couple

$$-4\pi\mu bv/(K_0 + \tfrac{1}{2}),$$

which is the value of the couple on the helix for a velocity $b\omega$ with $\Omega = 0$.

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